XXXV. *A Memoir concerning the moſt advantageous Conſtruction of Water-Wheels,* etc. *by Mr.* Mallet *of* Geneva. *Communicated by* M. Maty, *M. D. Sec. R. S. Tranſlated from the* French, *by* J. Bevis, *M. D. R. S. S. Read* March 26, 1767.

§ I. THE ſtream of rivers is of ſuch importance in moving machines of all kinds, that any attempt towards perfecting this part of mechanics may be conſidered as of very great utility.

The firſt wheel on which the ſtream acts is one of the moſt eſſential members of the machine, and it is eaſy to diſcern that the greater or leſs effect thereof muſt depend, in a great meaſure, on the manner of conſtructing this wheel, and on the dimenſions given to it. I ſhall not at preſent inquire whether wheels of different conſtructions from thoſe which have been long in uſe might be advantageouſly ſubſtituted in their ſtead; but confine myſelf, in this eſſay, to an examination of the moſt common ones, and to diſcover the means by which they may be made to produce the greateſt poſſible effects. Their conſtruction is very ſimple; they conſiſt of ſeveral planes inſerted into the ſame axle placed horizontally above the ſurface of the water, and in a poſition perpendicular to the ſtream. Theſe planes, called *float-*

Fig. 1.

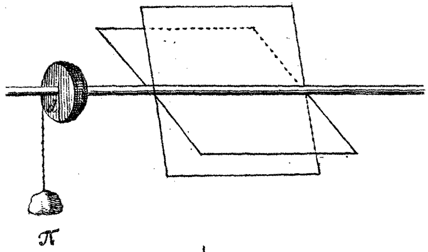


Fig. 2.

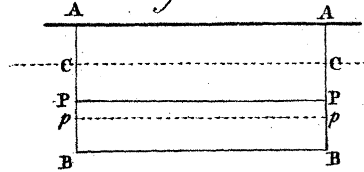


Fig. 3.

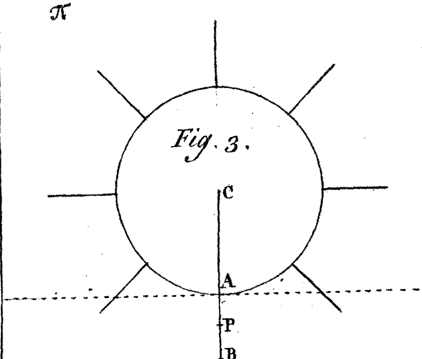


Fig. 4.

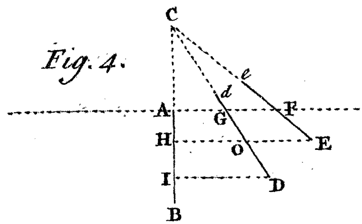


Fig. 5.

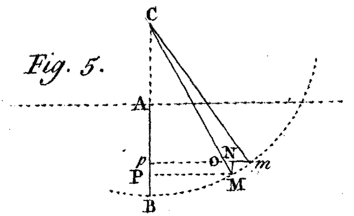


Fig. 6.

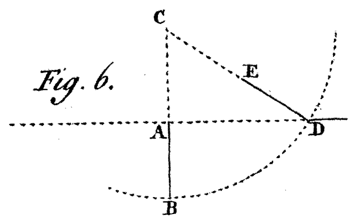
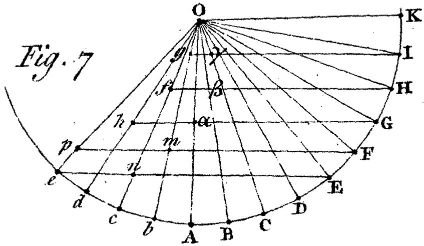


Fig. 7



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float-boards, by yielding to the action of the stream, cause the axle on which they are fixed to turn round, by means of several wheels, which take into each other, and give motion to the part destined to produce some purposed effect, as the mill-stone in a corn-mill.

The size of the float-boards, the velocity with which the wheel is to turn, and the number of the float-boards to produce the greatest possible effect, are three main things I propose to examine in the following inquiry.

In the first place I will suppose the total resistance which this wheel has to encounter, on the part of the machine, and which hinders it from moving so swift as the stream, to be expressed by a weight π, suspended to the extremity of a cord fixed to the circumference of a wheel whose radius is d, and which has the same axle as the float-board wheel, so that the effect of the stream is to raise the said weight π, as expressed in TAB. XV. *fig.* 1. I will likewise suppose, that the stream, by its velocity, moves through v feet in one second of time, and that this velocity is the same, though at different depths.

§ II. After these suppositions, the first thing that presents, is to determine what should be the size of the float-boards for the stream to be capable of raising the weight π with a certain determinate velocity.

Let A A B B, *fig.* 2. be one of the float-bords let into the axle A A, and placed vertically in the water, so as to receive the perpendicular impulse of the stream. Its horizontal length $BB=\beta$ feet, its vertical height $AB=\alpha$ feet, the velocity of the wheel

wheel at the point B, ſuch that it ſhall run through z feet in a ſecond; n pounds the weight of a cubic foot of water; and I will ſuppoſe the impulſe of the ſtream on a plane perpendicular to it is (as Dr. *Daniel Bernoulli* has ſtated in his *Hydrodynamica)* equal to the weight of a priſm of water, whoſe baſe is the plane, and its altitude the generating height of the velocity with which the plane is impelled. This being ſuppoſed; let AP $=x$, Pp, its differential, $=dx$, which will give the velocity of the float-board at the point $p=\frac{x}{a}z$, and the relative velocity of the ſtream with which the plane is impelled at the ſame point $= v-\frac{x}{a}z$ whoſe generating height is $\frac{(v-\frac{x}{a}z)^2}{60}$ feet, whence we have the weight of the parallelopiped of that height, and of the baſe PP pp equal to $\frac{n\beta dx}{60}(-v\frac{x}{a}z)^2$ pounds, which weight multiplied by the length AP (x) of the lever which tends to turn the plane, will give $\frac{n\beta x dx}{60}(v-\frac{x}{a}z)^2$ for the total effect of the ſtream on the little rectangle PP pp, whoſe integral is $\frac{n\beta}{60}\int.(vvxdx-\frac{2vz}{a}xxdx+\frac{zz}{aa}x^3dx)=$ $\frac{n\beta}{60}\left(\frac{1}{2}vvxx-\frac{2}{3}vz\frac{x^3}{a}+\frac{1}{4}zz\frac{x^4}{aa}-\frac{1}{2}vvff+\frac{2}{3}vz.\frac{f^3}{a}-\frac{1}{4}\frac{zzf^4}{aa}\right)$ (putting AC $=f$ for the diſtance between the axle and the ſurface of the water when the float-board has only its part CB plunged in the water) which (putting $x=a$) will become $\frac{n\beta}{60}\left(\frac{1}{2}vv-\frac{2}{3}vz+\frac{1}{4}zz.aa-\frac{1}{2}vvff+\frac{2}{3}\frac{vzf^3}{aa}\right.$

3 $-\frac{1}{4}$

$-\frac{1}{4}\frac{zzf^4}{\alpha\alpha}\Big)$ and will expreſs the effect on the whole plane CC BB, equal π d, the product of the weight π by the length d of the lever on which it acts in oppoſing the motion of the wheel.

§ III. If the wheel be plunged as deep as its axle, that is, $f = o$, the equation is changed into this $\frac{n\beta\alpha\alpha}{60}\left(\frac{1}{2}vv - \frac{2}{3}vz + \frac{1}{4}zz\right) = \text{d}\pi$, where it appears 1°. That the quantities d, π, v and z remaining the ſame, we have β inverſely proportional to the ſquare of α, whence it follows, that if the length β is to be diminiſhed without altering the effect of the float-board, the height α muſt be encreaſed proportionally to the ſquare root of β; for example, if β is to be made four times leſs, it will be ſufficient to double the height α. 2° That likewiſe the velocity of the float-board remaining the ſame, the weight π will be in the compound ratio of the length β, and of the ſquare of the height $\alpha\alpha$. 3° Without meddling with the dimenſions of the float-board, the more the quantity z is increaſed, the more muſt the weight π be diminiſhed. If z be made $= o$, we have $\pi = \frac{n\beta\alpha\alpha}{60\text{d}} \cdot \frac{1}{2}vv$, and if $z = v$ we have $\pi = \frac{n\beta\alpha\alpha}{60\text{d}} \cdot \frac{1}{12}vv$, that is ſix times greater than in the firſt caſe; which is very conformable to the nature of things, for when the wheel is in motion, the ſtream then not acting upon it but with the exceſs of its velocity above that of the wheel, it follows, that the greater ſuch velocity is, the more will the effect of the ſtream be diminiſhed.

It

It follows from our laſt remark, that the greateſt weight with which the ſtream can conſtitute an equilibrium, will be $= \frac{n\beta\alpha\alpha vv}{120d}$, but then the wheel will not have any motion, nor conſequently the weight π: If the float-board be increaſed, or the weight diminiſhed, from that inſtant the wheel will begin to turn, and the ſwifter as the float-board is greater, or the weight leſs; but in moſt machines, it is required that the weight may be the greateſt poſſible, as alſo the velocity with which it is raiſed. A queſtion therefore here offers itſelf, whoſe ſolution is of much importance. What muſt be the velocity of the float-board whoſe dimenſions are given, that the product of the weight by its velocity ſhall be the greateſt poſſible?

§ IV. The velocity of the weight π is $\frac{d}{\alpha}z$ feet in a ſecond, which being multiplied by the value of $\pi = \frac{n\beta\alpha\alpha}{60d}\left(\frac{1}{2}vv - \frac{2}{3}vz + \frac{1}{4}zz\right)$ will give the product $\frac{n\beta\alpha}{60}\left(\frac{1}{2}v^2z - \frac{2}{3}vzz + \frac{1}{4}z^3\right)$ which muſt be a *maximum*; for which purpoſe make $\frac{1}{2}vvdz - \frac{4}{3}vzdz + \frac{3}{4}z^2dz = 0$; whence we have $z = \frac{8 - \surd 10}{9} = 0,53752v$: this value of z being ſubſtituted, make the equation $\frac{1}{2}vv - \frac{2}{3}vz + \frac{1}{4}zz = \frac{11 + 2\surd 10}{81}vv$, ſo that we have the equation $\beta\alpha\alpha = \frac{11 + 2\surd 10}{4860} + \frac{d\pi}{nvv} = 280,529 + \frac{d\pi}{nvv}$, which expreſſes the dimenſions of the float-board where the effect will be the greateſt poſſible. If the float-board be plunged no deeper than to CC, as we have at firſt ſuppoſed, the moſt advantageous value of of z may be determined in the ſame manner, which will be found

$$z = \frac{8\alpha^4 - 8\alpha f^3 - \alpha\sqrt{10\alpha^6 + 54\alpha^4 f^2 - 128\alpha^3 f^3 + 54\alpha^2 f^4 + 10 f^6}}{9(\alpha^4 - f^4)}\,v$$

If $f = 0$, this value of $z = 0.537v$
$f = 0{,}25\,\alpha = \frac{1}{4}\alpha$. $= 0{,}498v$
$f = 0{,}3\,\alpha$ $= 0{,}486v$
$f = 0{,}5\,\alpha = \frac{1}{2}\alpha$. $= 0{,}436v$
$f = 0{,}7\,\alpha$ $= 0{,}390v$
$f = 0{,}9\,\alpha$ $= 0{,}353v$
$f = \alpha$ $= 0{,}333v$

By the inſpection of theſe different values it appears that this value of z diminiſhes as the plunged part is greater, and that this velocity can never exceed the quantity $0{,}537v$, nor be leſs than $\frac{1}{3}v$ *.

This value of z and of its ſquare zz being ſubſtituted in the general formula (§ II.) we ſhall obtain from it the following equation:

$$\frac{60\,d\,\pi}{v\beta vv} = \frac{11\alpha^6 - 27\alpha^4 f^2 + 32\alpha f^3 - 27\alpha^2 f^4 + 11 f^6 + 2(\alpha^3 - \alpha ff)\sqrt{10\alpha^6 + 54\alpha^4 f^2 - 128\alpha^3 f^3 + 54\alpha^2 f^4 + 10 f^6}}{81(\alpha^4 - f^4)}$$

which for a given relation between f and α will ſhew the breadth β for producing the greateſt effect.

* If in the value of z we make $f = \alpha$, we have $z = \frac{0}{0}$ which obliges us to take, according to the common method, the differentials of the numerator and of the denominator, conſidering f as variable, and the relation of theſe differentials will give the value of z: but on account of the radical quantity, the calculus being ſomewhat tedious, and, again bringing out $z = \frac{0}{0}$ and that after ſeveral ſimilar operations, it is better to have recourſe to the equation from which the value of z was deduced; this equation is $\frac{3}{4}\alpha v z.\frac{\alpha^3 - f^3}{\alpha^4 - f^4} - \frac{1}{2}\alpha\alpha vv.\frac{\alpha^2 - f^2}{\alpha^4 + \ + f^4}$, which by the above operation will be $\frac{3}{4}zz = vz - \frac{1}{4}vv$, and $z = \frac{1}{3}v$.

 As

As the extremity of the float-board muſt have a certain velocity depending on the relation of the height α to the plunged part, and as the velocity of the weight $\pi = \frac{d}{\alpha} z$, it follows that if we would increaſe the velocity of π, we muſt diminiſh the height α and increaſe the breadth β, ſo that the product $\beta\alpha\alpha$ and the relation of f to α may be the ſame as before; for example, if the wheel be plunged as deep as the axle, to double the velocity of the weight, the height of the float-board muſt be reduced one half, and its breadth be quadrupled.

§ V. It may ſo happen that the channel on which the wheel is placed ſhall be ſo ſhallow and narrow, as not to allow the float-boards the neceſſary dimenſions, for raiſing the weight with a convenient velocity. In this caſe we are obliged to raiſe the axle of the wheel above the ſurface of the water, ſo much that the lever on which the ſtream acts may be long enough to recompenſe the ſmallneſs of the float-boards. Herein it is neceſſary to ſolve the following problem:

The breadth b, *and the height* a, *of the float-board* AB *being given; to find the radius* CA (r) *of the wheel which ſhall cauſe the weight* π *to aſcend with the velocity* $\frac{d}{r + \alpha} z$.

The exact ſolution of this problem might be deduced from the formula (§ II.) which would render the operation tedious, the equation being of the fourth degree; but it may be rendered far more ſimple by a ſuppoſition which is but little wide of the truth when AB is but ſmall in compariſon of C A, and this is to conſider confider all the points of the float-board AB as affected with the fame velocity z.

Let CP fig. 3. $= x$, we fhall have $\frac{nb}{60}(v-z)^2 \int . x\,dx$ for the effect of the portion AP, and $\frac{nb}{60}(v-z)^2 \times \frac{2ar+aa}{2}$ for the effect of the whole float-board AB. This quantity muft be made equal to dπ, and then, juft as in the foregoing cafes, fuch a value of z be fought, that the weight π and its velocity may be the greateft poffible; that is, the differential of $z\,(v-z)^2$ muft be made $=$ o, which gives $z = \frac{1}{3}v$. Therefore d $\pi = \frac{nabvv}{270}(2r+a)$, and $r = \frac{270\,d\pi - naabvv}{2nabvv}$, and the velocity of the weight π will be $= \frac{2nd\,abvv}{810\,d\pi + 3nbaav} \times v$.

§ VI. We have feen that the calculus was much fimplified by fuppofing one of the velocities conftant for all points of the float-board. For this velocity being c, the effect of the whole float-board will be fimply $\frac{n\beta}{60}(vc)^2 \frac{\alpha\alpha - ff}{2}$. It will therefore not be unufeful to inquire what this velocity c muft be, that the effect of the float-board may be the fame, as fuppofing, as we have hitherto done, a variable velocity, and proportional to the diftances from the axle, we have only to make $\frac{n\beta}{60}\overline{(v-c}^2\left(\frac{\alpha\alpha - ff}{2}\right)$ $= \frac{n\beta}{60}\left(\frac{1}{2}vv\,(\alpha\alpha - ff) - \frac{2}{3}vz\frac{\alpha^3 - f^3}{\alpha} + \frac{1}{4}zz\frac{\alpha^4 - f^4}{\alpha\alpha}\right)$ (§ IV.) but the equation whence we got the value of z

 (§ IV.)

(§ IV.) $\frac{1}{2} vv - \frac{4}{3} \frac{vz}{\alpha} . \frac{\alpha^3 - f^3}{\alpha^2 - f^2} + \frac{1}{4} zz . \frac{\alpha\alpha + ff}{\alpha\alpha} = 0$, we shall have $c = v - \sqrt{\frac{1}{2} vv - \frac{1}{4} zz . \frac{\alpha\alpha + ff}{\alpha\alpha}}$.

If $f = 0$, we have $zz = 0{,}288 vv$ and $c = 0{,}345 v$
$f = \frac{1}{2} \alpha$ $zz = 0{,}190 vv$ $c = 0{,}336 v$
$f = \alpha$ $zz = \frac{1}{9} vv$ $c = \frac{1}{3} v$.

so that whatever be the relation of f to α, the velocity c is ever nearly $= \frac{1}{3} v$, and the more exactly so, as f is greater. Wherefore we may always assume $\frac{p}{270} n \beta vv . (\alpha\alpha - ff) = d\pi$ for the effect of the stream upon a float-board whose plunged part is $\alpha - f$; this effect will be increased in the ratio of 4 to 9, when the wheel has no motion, for making $c = 0$, we find it $= \frac{1}{120} . n \beta vv . (\alpha\alpha - ff)$.

§ VII. Hitherto we have all along supposed, that the float-board did through its whole plunged part receive the perpendicular impulse of the stream; but it is easily understood, that the wheel coming to turn, presents to the stream the plane of the float-board under an angle which is continually varying, which diminishes its effect every instant, as it removes from the vertical: This inconvenience may be remedied by multiplying the number of the float-boards, so that when the first is removed from the vertical as far as a certain point, the next may occupy that advantageous place, to be in its turn replaced some time after by a third, and so on. Now our third inquiry is, to assign the angle contained between two float-boards, or, which comes to the same, the number of float-boards the wheel should consist of, that its effect may be the greatest possible, being of no less importance than the preceding ones. To begin then with the most simple

ſimple caſe; we will ſuppoſe the wheel immoveable, or that $c = o$, and proceed to inveſtigate, whether, ſuppoſing the number of float-boards to be greater, the ſum of the effects will come out greater or leſs than what reſults from one ſingle float-board placed vertically.

In order to a general ſolution of this queſtion, we will ſuppoſe two float-boards CD and CE fig. 4. making any angles with the vertical, and let us compare the effect of the ſingle float-board GD with the effect reſulting from the float-boards FE and GD taken together, which will be reduced to FE and OD, becauſe the part OG becomes uſeleſs, as the ſtream is intercepted by FE. Let $CB = CD = CE = a$, $CA = f$, coſin. $BCD = m$, coſin. $BCE = \mu$, which gives $CG = \frac{f}{m}$ $CF = \frac{f}{\mu}$ and $CO = \frac{\mu}{m} a$. Then we ſhall find, by § VI, the effect of $GD = \frac{n\beta vv}{120} (mmaa - ff)$ that of $OD = \frac{n\beta vv}{120} (mmaa - \mu\mu aa)$. and that of $FE = \frac{n\beta vv}{120} (\mu\mu aa - ff)$; whence it appears, that the ſum of the two laſt is exactly equal to the firſt, which will ever hold good whatever be the value of f.

Whence ariſes the following theorem:

Whether the wheel be plunged quite up to the axle, or only in part ſo, provided it be immoveable, and that one of its float-boards be placed vertically, its effect will be conſtantly the ſame, whatever be the number of float-boards oppoſed to the ſtream, even though it were infinite.

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The latter part of this theorem, though flowing from the general demonſtration, may be alſo demonſtrated, immediately, in the following manner; Let BP fig. 5. be $= x$, we have MO $= \frac{\alpha dx}{\alpha - x}$, and CO $= \frac{\alpha\alpha - dx - \alpha x}{\alpha - x}$; $\overline{CO}^2 = \frac{\alpha^4 - 2\alpha^3 x + \alpha^2 x^2 - 2\alpha^3 dx + 2\alpha\alpha x dx}{\alpha\alpha - 2\alpha x + xx}$ neglecting the dx^2) and $\alpha\alpha - \overline{CO}^2 = \frac{2\alpha\alpha x dx}{\alpha - v}$; Therefore the effect of the ſtream upon OM, which is $= \frac{n\beta vv}{120} \cdot \left(\alpha\alpha - \overline{CO}^2\right) \frac{\overline{CP}^2}{\overline{CP}^2}$ will become $= \frac{n\beta vv}{120}$ $(2\alpha x - 2x dx)$ whoſe integral is $= \frac{n\beta vv}{120}$ $(2\alpha x - xx)$ wherein puttting $x = \alpha - f$, we have $\frac{n\beta vv}{120}$ $(\alpha\alpha - ff)$ for the total effect of the ſtream upon the wheel, which is the ſame as that of a ſingle float-board A B in a vertical poſition.

§ VIII. This theorem will alſo hold true for the caſe of § V. wherein we have ſuppoſed the height of the float-boards very ſmall, in compariſon of the radius of the wheel; we have ſeen that the effect of a ſingle float-board placed vertically was $= nabvv$ $(2r + a)$; the demonſtration of the preceding § will be applicable here after the ſame manner, and will ſhew that whatever be the number of float-boards, the effect will be ever the ſame.

It does not however follow that the number of float-boards ſhould be indifferent; for the wheel coming to turn the float-board, its lower part, which received the perpendicular impulſe, will no longer receive it otherwise than obliquely, and the effect will diminish till the angle formed by two neighbouring float-boards be bisected exactly by the vertical, which will render the first entirely useless; after which the effect will increase anew, and will become again greatest, when the second float-board is got to the vertical; so that in order to fix upon the most advantageous number of float-boards, regard must be had to the sum of the different effects for all the situations of the float-boards during one whole turn of the wheel.

Whence it follows that in this case, wherein they are supposed very small, the greater their number is, the greater will be the sum total of the effects, since, if that number were infinite, there would be a float-board in a vertical position every instant.

§ IX. This will no longer hold good, if the height of the float-boards be more considerable, and it be found necessary to take the different velocity of their different points into consideration; by comparing (fig. 4.) the pressure on FE with that on the portion GO, they will be found no longer equal, as in the foregoing case; it is true that the same quantity of fluid acts on these two planes, and the disadvantage which FE has by receiving the impulse more obliquely, is exactly compensated, as before, by the length of the lever, but the difference arises from the different velocity of the corresponding points of FE and GO; those velocities are in the ratio of CF to CG, or as cos. ACG to cos. ACF, which shews that the effect of FE is always less than that of GO, and consequently the effect must be diminished, by adding a greater number of float-boards;

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the said effect will be greatest when there is only one float-board placed vertically, and least when their number is infinite: let us enquire what it will be in this latter case. We will suppose the same fig. 5, and the same denominations as in § VII. We had $CO = \frac{\alpha\alpha - \alpha x - \alpha dx}{\alpha - x}$, we shall have (neglecting dx^2, dx^3, and dx^4) $\alpha\alpha - \overline{CO}^2 = \frac{2\alpha\alpha dx}{\alpha - x}$, $\frac{\alpha^3 - \overline{CO}^3}{\alpha}$, and $\frac{\alpha^4 - \overline{CO}^4}{\alpha\alpha} = \frac{4\alpha\alpha dx}{\alpha - x}$. Now the pressure on OM is, by § II, $= \frac{n\beta}{120} \frac{\overline{\alpha - x}^2}{\alpha\alpha} \left(vv(\alpha\alpha - \overline{CO}^2) - \frac{4}{3} vz . \frac{\alpha^3 - \overline{CO}^3}{\alpha} + \frac{1}{2} zz . \alpha^4 - \frac{\overline{CO}^4}{\alpha\alpha} \right)$ which (by putting for CO its value) will become $= \frac{n\beta}{120} (2\alpha dx - 2x dx) (v - z^2)$ whose integral $\frac{n\beta}{120} (2\alpha x - xx) (v - z^2) = \frac{n\beta (v - z^2)}{120} (\alpha\alpha - ff)$ (making $x = \alpha - f$) will express the effect resulting from an infinite number of float-boards: this least effect will be to the greatest, that is when there is but one float-board, as $(v - z^2) : vv - \frac{4}{3} \frac{vz}{\alpha} . \frac{{}^3 - f^3}{\alpha\alpha - ff} + \frac{1}{2} \frac{zz}{\alpha\alpha} . (\alpha\alpha + ff)$ or as $(v - x^2) : \frac{1}{2} vv - \frac{1}{4} zz . \left(\frac{\alpha\alpha + ff}{\alpha\alpha}\right)$ §. VI.

This ratio will be that of	1 : 2	if $f = 0$
	1 : 1,485	$f = \frac{1}{2}\alpha$
	1 : 1	$f = \alpha$

§ X. If

§ X. If we take nothing but the moſt advantageous poſition into conſideration, and preſerve the greateſt effect entire, it follows that the angle BCD (fig. 6.) between two float-boards muſt be ſuch that E ſhould enter the water at the inſtant when AB quits the vertical, ſo that the coſine of that angle be $= \frac{f}{\alpha}$; in conſequence of which the following table may be conſtructed, ſhewing what the number of float-boards ſhould be for a given ratio between f and α.

For 4 float-boards, we have $f =$	0,
5	$0{,}3090\alpha$
6	$0{,}5000\alpha$
7	$0{,}6236\alpha$
8	$0{,}7071\alpha$
9	$0{,}7660\alpha$
10	$0{,}8090\alpha$
12	$0{,}8669\alpha$
14	$0{,}9009\alpha$
16	$0{,}9239\alpha$
18	$0{,}9397\alpha$
20	$0{,}9510\alpha$
&c.	&c.

§ XI. Certain authors treating of hydraulics, have in this part thereof given the ſame table, as containing the true number of float-boards the wheel ſhould conſiſt of: but we have ſeen upon what principle it was formed, and that it was only to preſerve entirely the effect of the vertical float-board; from whence it follows not that the number of float-boards which it aſſigns ſhould be the moſt advanta-

 geous.

geous. To which purpose the effect produced from every position of the wheel, and for the different number of the float-boards, should be computed; the number which gives the arithmetical mean between all these effects, the greatest of all, will be that to be chosen, and preferred before what the above table indicates.

It may be sufficiently satisfactory to compute only the effect from 1 to 10 degrees. Thus, for example, for the wheel entirely plunged, we are to find the effect (fig. 7.)

1° on OA, 2° on OI and *gb*, 3° on OH, and *fc*, 4° on OG and *hd*, 5° on OF, and *pe*, 6° on OE, 7° on OD, 8° onOC, and 9° on OB.

After which the wheel returns into the same position it had at first; and we are to divide the sum of all these effects by 9, to get the arithmetical mean.

We will next suppose the number of six float-boards for the same case of $f = 0$, and compute the following effects.

1° on	$OG + aA$,	3° on	$OE + nc$,	5° on	$OI + \gamma C$
2°	$OF + mb$,	4°	OD	6°	$OH + \beta B$.

The sum of all these effects divided by 6 will give the effect of the wheel of 6 float-boards.

The same thing, supposing the angle 40 degrees, or 9 float-boards, and as after a revolution of these 40 degrees, the wheel returns into a similar position, the same must be divided by 4.

Then for an angle of 30 degrees we are to divide by 3, and so on.

I have made this computation to great exactness, for the case of $f = 0$, $f = \frac{1}{2}\alpha$, and $f = 0,866\,\alpha = \alpha$ cos. 30°; the result,

1° If $f = 0$.

for 4 float-boards, the arith. mean $= 0.335$ ($\frac{1}{10}$ $n\beta\alpha\alpha vv$).

It may be observed in this first case, that there is some advantage in taking 6 float-boards instead of 4 shewn by the table; the effect will be increased in the ratio of 100 to 118, and yet will be more than about $\frac{92}{100}$ of the greatest effect above calculated for a single vertical float-board; so that the found dimensions must be a small matter altered, and the quantity $\beta\alpha\alpha$ increased by $\frac{1}{10}$.

for 6	= 0,396 (&c.
9	= 0,336 (&c.
12	= 0,323 (&c.
18	= 0,295 (&c.
an infinite number	= 0,214 (&c.

2° If $f = \frac{1}{2}\alpha$.

for 6 flo.	0,277 (&c.
7	0,281 (&c.
9	0,285 (&c.
12	0,284 (&c.
18	0,276 (&c.
an infinite number	0,238 (&c.

In this second case 9 float-boards are to be taken instead of 6 shewn by the table, though the difference will be but very small; and we shall have an effect which will be $\frac{86}{100}$ of that of a vertical float-board, and in that ratio that the quantity $\beta\alpha\alpha$ found by the above formulæ, must be increased.

3° If $f = 0{,}866\,a$

for 12 fl.	= 0,099 (&c.
18	= 0,099 (&c.
36	= 0,104 (&c.
an infinite number	= 0,103 (&c.

In this third caſe the difference is ſtill very ſmall, and the effect reſulting from 36 float-boards will be $\frac{03}{100}$ of the effect of a ſingle vertical float-board.